Indian J. Phys. 67B (2), 163-175 (1993)

Some structural features and the equilibrium of finite two-component non-rotating isothermal spheres

J P Sharma and R B Yadav*
Department of Applied Sciences, M. M. M. Engineering College, Gorakhpur-273 010, Uttar Pradesh, India

Received 7 November 1991, accepted 15 January 1993

Abstract : Analytic structural features of finite (confined) isother mal spheres composed of particles of two different masses have been discussed. Approximate analytical solutions to the equilibrium equations are obtained in concise and simple from useful for short computer program or on small calculator. The qualitative behaviours of the physical quantities $\rho_1(r)/\rho_1(o)$, $\rho_2(r)/\rho_1(o)$, $P/K_BTn_1(o)$, $\rho/m_1n_1(o)$, $p/m_1n_1(o)$, $\xi^2\psi'$, ψ' and ψ for two component isothermal models with $\mu = 2$, 5 and 8, $\lambda = 1$ are shown.

Keywords : Isothermal spheres, structural features, analytic structure

PACS Nos. : 04.60.+n, 97.10.Cv

1. Introduction

Study of the structure of self-gravitating polytropic and isothermal gas spheres in Newtonian theory dates back to early twentieth century [1-4]. The self gravitating polytropic and isothermal sheets, such as saturn ring system and the Laplacian-disk cosmogonies have been considered by several authors [5-7]. Some new ideas to the study of polytropic and isothermal cylinders have been added by some researchers [8-10] and the author [11, 12] who tackled the problem of polytropic and isothermal plane-symmetric configurations from view point of the ease with which equilibrium equations can be solved.

Building of gas sphere models (stars) follows the traditional work in theoretical astrophysics. It helps to understand the contacts between gravitation theory and reality (experiments and observations). Isothermal spheres are of interest to astrophysicists because of their roles in the studies of stellar structure, star formation, cluster dynamics and galactic dynamics. Considerable amount of work has been done in both non-relativistic [4, 11, 13-16] (Newtonian theory) and relativistic [17, 18] regimes

* Working as Research Assistant supported by a grant fomn CST, UP, Lucknow, India.

The dynamical evolution of spherical (one-component) systems has been studied in detail by several authors. Spitzer and Hart (hereafter called SH) [19] extended the theory to systems with a number of masses who [20] had previously pointed out that with the addition of even a single component with a different particle mass may have a considerable effect on the structure and stability of the system. This instigates us to study the analytical structure of finite two-component isothermal (TCI) spheres by Pade (2, 2) approximation method different from previous numerical procedures. The main advantages of our analytical study of the TCI spheres are : Firstly, it is quite economical to build up by simply taking intó consideration the variations in physical parameters (which could have been probably costlier for such experimentation by following stellar dynamics numerical calculations). Secondly, sturcture formulae are obtained in simple and concise form useful for very short computer programs or even useful for small electronic pocket-calculators, and (iii) it provides a direct comparison with evolutionary numerical calculations of Chandrasekhar's one-component case [4].

Thus in the present work, we study the analytic equilibrium structure of two-component (finite) isothermal configurations; stability considerations of these objects, anticipated to be a richer problem in itself would, however, be postponed for future work. Section 2 deals with the equilibrium structure (eq. (6)). Approximate analytical solution to the structure equations are given in Section 3.

2. Equilibrium structure equations

For the system (particles of mass m_i of phase-space density f_i (r, p) with stationary entropy S with respect to arbitrary infinitesimal variations in f_i subject to the conditions that the total energy E and the total numbers N_i of particles of species i remain fixed), we have, in equilibrium [21],

$$f_i = \exp\left[\frac{\mu_i}{K_B T} - \frac{1}{K_B T}\left(\frac{p^2}{2m_i} - m_i \varphi - 1\right)\right], \tag{1}$$

where the gravitational potential $\varphi(r)$ is defined by

$$\varphi(r) \equiv G \sum_i \frac{m_i f_i\, d^3r'\, d^3p'}{|r - r'|\,\Delta}; \tag{2}$$

$B = -\frac{1}{T}$ (temperature), $\alpha_i = \frac{\mu_i}{T}$ (μ_i = chemical potential for species i). The equation of equilibrium is given by

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -4\pi G \sum_i m_i n_i(\mathrm{o}) \exp \frac{m_i}{K_B T}\left[\varphi(r) - \varphi(\mathrm{o})\right], \tag{3}$$

where the number density of particles of species i is expressed by

$$n_i(r) \doteq n_i(\mathrm{o}) \exp \frac{m_i}{K_B T}\left[\varphi(r) - \varphi(\mathrm{o})\right]. \tag{4}$$

Define the two dimensionless variables ξ and ψ by

$$\psi = \frac{m_i}{K_B T}[\varphi(r) - \varphi(o)], \; r = \alpha^{-1}\xi; \; \alpha = [4\pi G\, m^2{}_1 n_1(o)/K_B T\,]^{1/2}. \tag{5}$$

In particular case, for the system composed of two species of particles only, we have from eqs (3) and (5)

$$\frac{d^2\psi}{d\xi^2} + \frac{2}{\xi}\frac{d\psi}{d\xi} - (e^{-\psi} + \lambda\mu\, e^{-\mu\psi}) = 0; \tag{6}$$

$$\mu = m_2/m_1 \geq 1, \;\; \lambda \equiv n_2(o)/n_1(o) \geq 0. \tag{7}$$

Eq. (6) is analogous to the equation of hydrostatic equilibrium of Chandrasekhar's [4] one—component case ($\lambda = 0, \mu = 1$). We seek in Section 3, approximate analytical solution of (6) satisfying the initial conditions

$$\psi(o) = 0, \;\; \psi'(o) = 0 \; (\zeta' \equiv \frac{d}{d\xi}) \tag{8}$$

3. Structure of Two-component isothermal spheres; numerical results, and physical significance of structural parameters

It is possible to discuss the physical properties of two-component isothermal spheres (TCIS) provided solutions to the structural eq. (6) are known. To this aim, one may take resort to the standard numerical methods, for example, R—K method, variational technique, etc, when, preferably closed form solutions are not available. In the former case, as here, the calculations might involve mathematical complexities, and might be inconveniently lengthy. Hence from view point of astrophysical applications, certain computational difficulties could be avoided by means of a suitable series expansion of the function $\psi(\xi)$ valid as $\xi \to 0$.

(a) Approximate analytical solutions of eq. (6) :

The series solutions of eq. (6), including terms upto ξ^{10}, in the neighbourhood of the origin ($\xi \to 0$), satisfying the initial conditions in (8), can be assumed as

$$\psi(\xi) = a\xi^2 + b\xi^4 + c\xi^6 + d\xi^8 + e\xi^{10} + \ldots\ldots \tag{9}$$

With the help of eqs. (6) and (9), the coefficients a, b, c, d and e may be found by straightforward means, *i.e.* we may obtain :

$$a = \frac{1+\lambda\mu}{6}, \;\; b = -\frac{1+\lambda\mu^2}{20}a, \tag{10}$$

$$c = \frac{-b(1+\lambda\mu^2) + Aa^2(1+\lambda\mu^3)}{42},$$

$$d = \frac{1}{72}[-c(1+\lambda\mu^2) + 2Aab\,(1+\lambda\mu^3) - Ba^3\,(1+\lambda\mu^4)],$$

$$e = \frac{1}{110}[-d(1+\lambda\mu^2) + A(b^2+2ac)(1+\lambda\mu^3) - 3Ba^2b(1+\lambda\mu^4)$$

$$+ Ca^4(1 + \lambda\mu^5)],$$

$$A = \frac{1}{2!}, \; B = \frac{1}{3!}, \; C = \frac{1}{4!},$$

where if we put $\lambda = 0$, $\mu = 1$, we may recover one component case [4].

Now we may express the series in (9) as a Pade (2, 2) approximation in the form of a rational function

$$\psi(\xi) = \psi_{22}(\xi) = \frac{1 + A'\xi^2 + B'\xi^4}{1 + C'\xi^2 + D'\xi^4} a\xi^2, \tag{11}$$

where $A' = a_1 + C'$, $B' = a_2 + a_1C' + D'$;

$$C' = (a_2a_3 - a_1a_4)/\Delta, \; D' = (a_2a_4 - a_3^2)/\Delta;$$

$$\Delta \equiv a_1a_3 - a_2^2$$

$$a_1 = b/a, \; a_2 = c/a, \; a_3 = d/a, \; a_4 = e/a.$$

In Figures 1, 2, 3, 4, 5, 6, 7 and 8 we plot $(\rho_1(r)/\rho_1(o), \xi)$, $(\rho_2(r)/\rho_1(o), \xi)$, $(P/K_BTn_1(o),$ $(\rho/m_1n_1(o), \xi)$, $(\bar{\rho}/m_1n_1(o), \xi)$, $(\xi^2\psi', \xi)$, (ψ', ξ) and (ψ, ξ) curves, respectively, for fixe concentration $\lambda = n_2(o)/n_1(o)$.

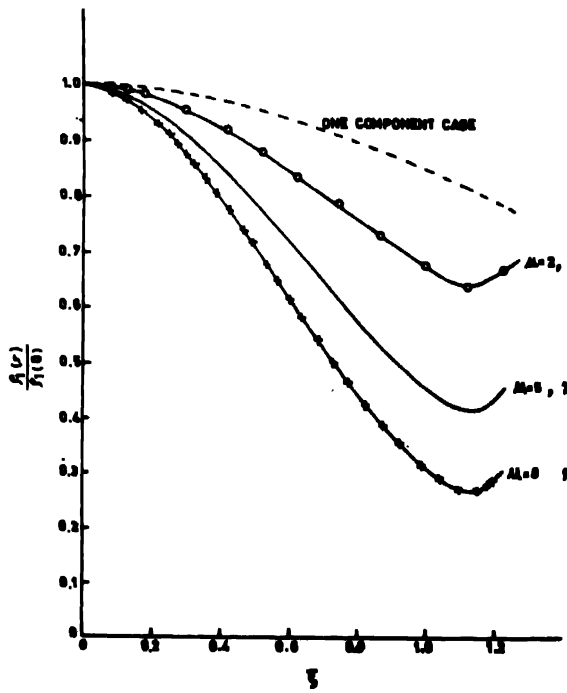

Figure 1. The mass density distribution $\rho_1(r)$, measured in units of $\rho_1(o)$, plotted as a function of dimensionless variable ξ, for particles of low mass component m_1 for systems with $\mu = 2(3)8$, $\lambda = 1$. The dashed curve represents one-component system [4].

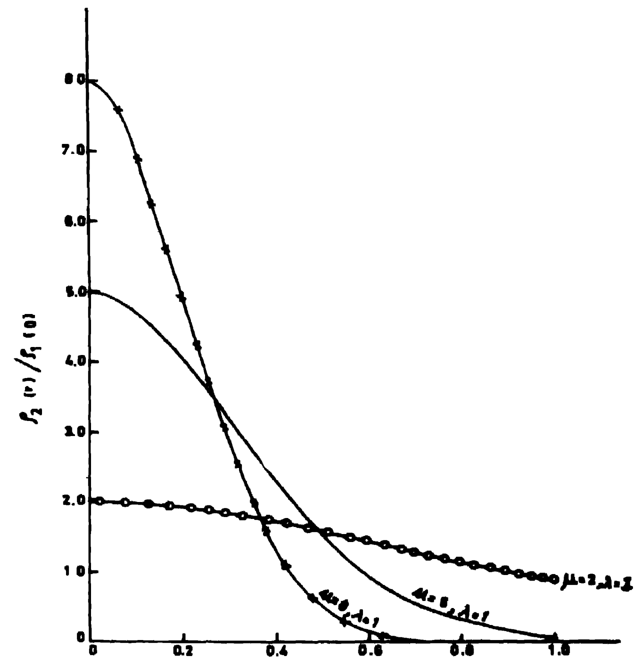

Figure 2. The mass density distribution $\rho_2(r)$, measured in units of $\rho_1(o)$, plotted as a function of dimensionless variable ξ, for particles of heavy mass component m_2 with $\mu = 2(3)8$, $\lambda = 1$.

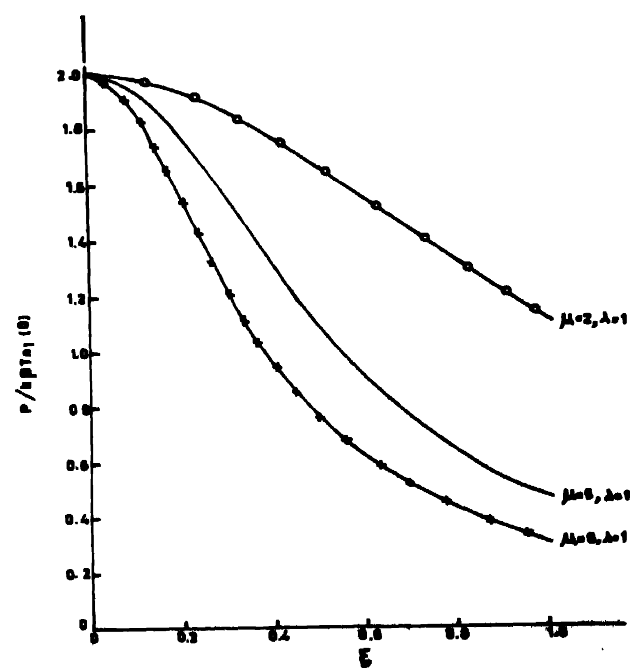

Figure 3. The pressure distribution $p(r)$, measured in units of $K_B T\, n_1(o)$, versus ξ, for two-component isothermal spheres. The curves for $\mu = 2(3)8$, $\lambda = 1$ are shown.

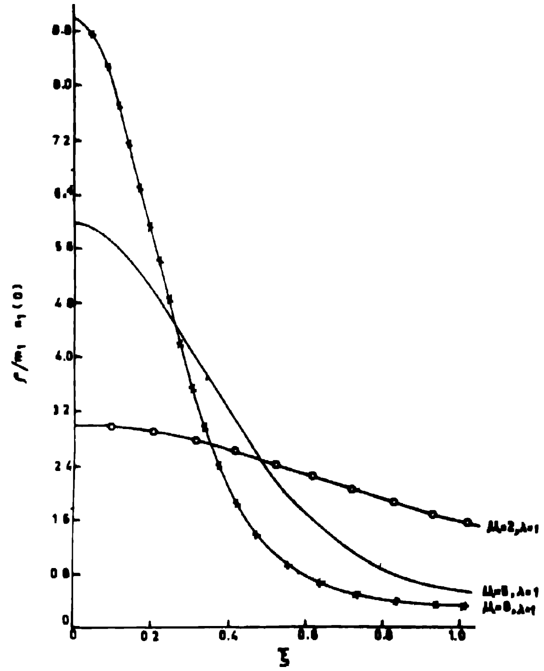

Figure 4. The mass density distribution $\rho(r)$, measured in units of $m_1 n_1(\mathrm{o})$, versus ξ, for two-component isothermal spheres. The curves for $\mu = 2(3)8$, $\lambda = 1$ are shown.

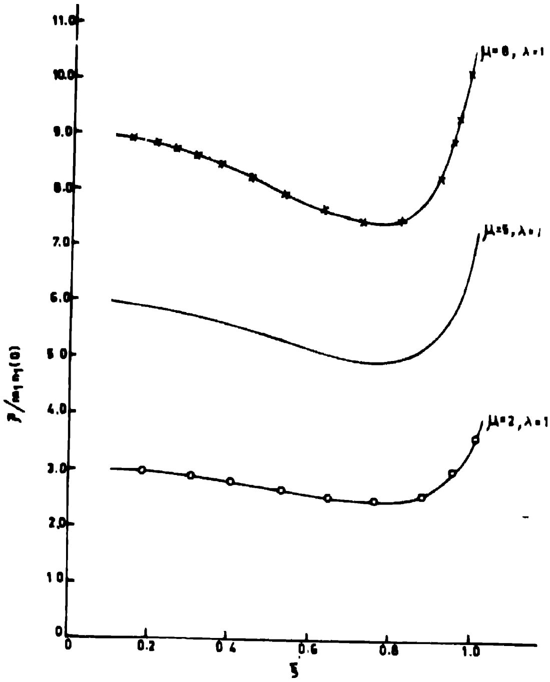

Figure 5. Properties of a two component system with $\mu = 2(3)8$, $\lambda = 1$. The mean (mass) density distribution $\bar{\rho}/m_1 n_1(\mathrm{o})$ versus ξ.

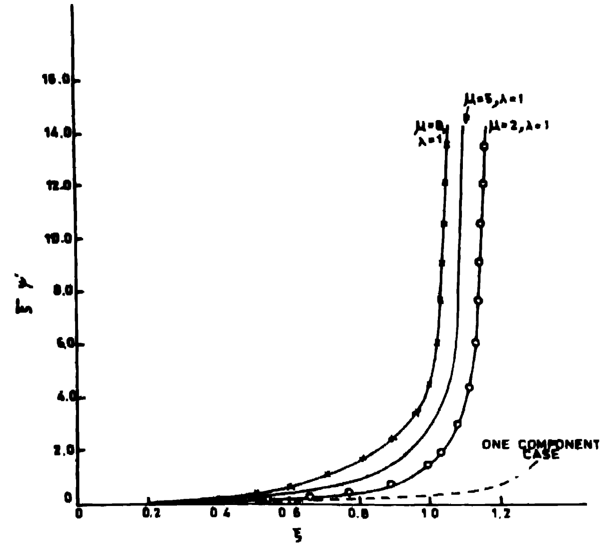

Figure 6. The quantity $\xi^2\psi'$ (proportional to the total mass) versus ξ for two-component isothermal spheres with $\mu = 2(3)8$, $\lambda = 1$.

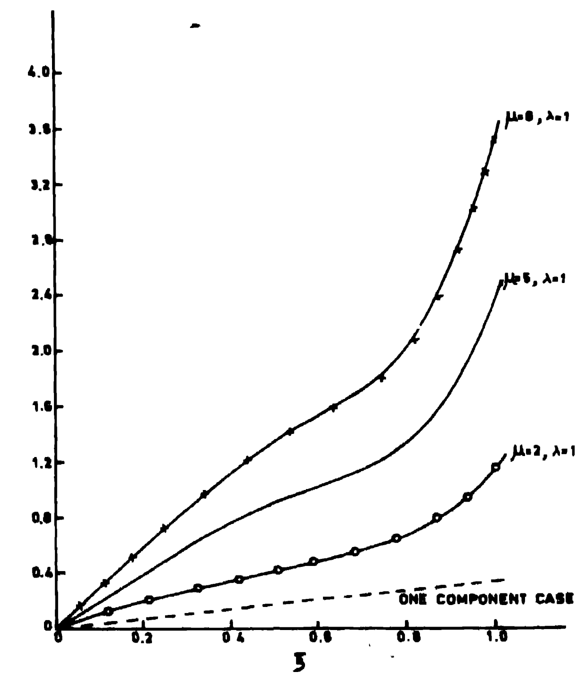

Figure 7. Properties of the derivative ψ' of the gravitational potential for the two-component isothermal spheres with $\mu = 2(3)8$, $\lambda = 1$, are compared with that of one-component case [4].

For comparison, one component case ($\lambda = 0, \mu = 1$) has been shown in Figures 1, 7 and 8. Figures 1 and 2 show that density distributions for both light (m_1) and heavy (m_2) mass particles fall off with increasing μ. The density distribution of the low mass (m_1) species, interior to a radius ξ, decreases with increasing μ for fixed concentration ratio $\lambda = 1$ and it shows a reverse trend with increasing μ for heavy mass (m_2) component, for small ξ (near the centre), but for larger μ, the picture is, however, not very much clear. For the two-component isothermal spheres, with μ = 2, 5, 8 and λ = 1, the pressure distribution P interior to a point ξ shows a decreasing trend with increasing μ (Figure 3). The mass density $\rho(\xi)/n_1(o)m_1$ for the TCIS, with μ = 2, 5 and 8, λ = 1 is shown in Figure 4. Figure 5 depicts the mass density (mean $\bar{\rho}/\rho_1$ (o) distribution for the TCIS with μ = 2, 5, 8; λ = 1. A comparison of our result between TCIS (μ = 2, 5, 8; λ = 1) and that of one-component case [4] is given in Figures 6, 7 and 8 where we plot, respectively, $\xi^2\psi'$ (proportional to total mass), ψ' and ψ (gravitational potential). It is interesting to note from Figures 6 and 7 that characteristic features of the system with μ = 2, λ = 1, correspond more closely to that of one-component case, for small ξ (near the centre).

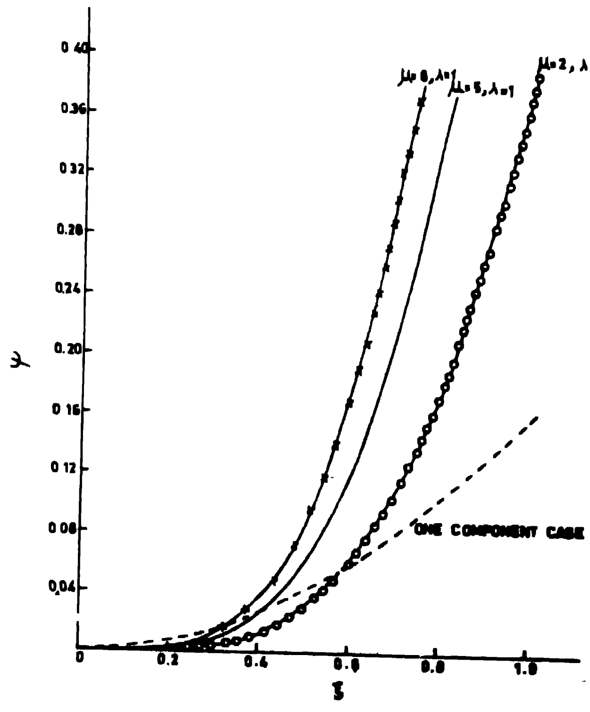

Figure 8. Properties of the dimensionless gravitational potential ψ for the two-component isothermal spheres with μ = 2(3)8, λ = 1, are compared with that of one-component sphere (dashed curve) [4].

(b) Approximate analytical solutions of eq. (6) in (U.V) phase plane :

We shall first consider the reduction of isothermal eq. (6) to an equation of the first order :

Define the two functions U and V related with the variables ξ and ψ by equations

$$U = \frac{(e^{-\psi} + \lambda\mu e^{-\mu\psi})}{\frac{d\psi}{d\xi}} = \frac{r}{M(r)} \frac{dM(r)}{dr},$$

$$V = \xi^{\psi} \frac{(e^{-\psi} + \lambda\mu e^{-\mu\psi})}{(e^{-\psi} + \lambda e^{-\mu\psi})} = \frac{r}{P(r)} \frac{dP(r)}{dr}, \tag{12}$$

where pressure P, mass M and the density ρ are given by equations :

$$P = K_B T [n_1(r) + n_2(r)] = K_B T n_1(\mathrm{o}) [e^{-\psi} + \lambda\mu e^{-\mu\psi}], \tag{13}$$

$$M = \int_{\mathrm{o}}^{r} 4\pi r^2 \rho \, dx = 4\pi\alpha^3 m_1 n_1(\mathrm{o}) \, \xi^2 \frac{d\psi}{d\xi},$$

$$\rho = m_1 n_1(\mathrm{o}) \exp \frac{m_1}{K_B T} [\varphi(r) - \varphi(\mathrm{o})] + m_2 n_2(\mathrm{o}) \exp \frac{m_2}{K_B T} [\varphi(r) - \varphi(\mathrm{o})]$$

$$= m_1 n_1(\mathrm{o}) [e^{-\psi} + \lambda\mu e^{-\mu\psi}].$$

Furthermore, with the help of eqs. (6) and (12), we obtain

$$\frac{1}{U(\xi)} \frac{dU(\xi)}{d\xi} = \frac{1}{\xi} [3 - U - VW] \tag{14}$$

$$\frac{1}{V(\xi)} \frac{dV(\xi)}{d\xi} = \frac{1}{\xi} [U + V(1 - W) - 1], \tag{15}$$

where $\quad W = (e^{-\psi} + \lambda\mu^2 e^{-\mu\psi})(e^{-\psi} + \lambda e^{-\mu\psi})/(e^{-\psi} + \lambda\mu e^{-\mu\psi})^2.$

Combining eqs. (14) and (15), we have

$$\frac{V}{U} \frac{dU}{dV} = \frac{3 - U - VW}{U + V(1 - W) - 1}. \tag{16}$$

We note here that like one-component case [4] ($W = 1$), it is not possible to express $\frac{dU(\xi)}{d\xi}$ (eq. (14)), $\frac{dV(\xi)}{d\xi}$ (eq. (15)), and $\frac{dU}{dV}$ (eq. (16)) in terms of U and V alone. The locus of points at which the solution curves have horizontal and vertical tangents are given by

$$U + VW = 3, \tag{17}$$

$$U + V(1 - W) = 1, \tag{18}$$

respectively. The point of intersection of the two loci are expressed by

$$U = 3 - \frac{2W}{2W - 1}; \quad V = \frac{2}{2W - 1}. \tag{19}$$

Furthermore, as $\xi \to 0$, we may obtain from expression in (12),

$$\left.\begin{aligned} U &\simeq 3\left(1-\frac{1}{3}\xi^2\right) \\ V &\simeq \frac{3}{2}\xi^2, \end{aligned}\right\} \quad (\lambda = 1, \mu = 2). \tag{20}$$

Clearly, therefore, as $\xi \to 0$, the curves pass through the point

$$U = 3, \quad V = 0, \quad (\xi = 0). \tag{21}$$

Similarly, from eqs. in (12), we have,

$$\left.\begin{aligned} U &\simeq 3\left(1-\frac{26}{15}\xi^2\right) \\ V &\simeq 6\xi^2 \end{aligned}\right\} \quad (\lambda = 1, \mu = 5). \tag{22}$$

$$\left.\begin{aligned} U &\doteq 3\left(1-\frac{13}{3}\xi^2\right) \\ V &\simeq \frac{27}{2}\xi^2 \end{aligned}\right\} \quad (\lambda = 1, \mu = 8). \tag{23}$$

Clearly, the (U, V) curves start at the point $(U = 3, V = 0)$.

To obtain approximate analytical solution of (16) in (U, V) plane, we proceed as follows : Let us assume a series expansion of expression in (16) in the form

$$U = 3 + \beta_1 V + \beta_2 V^2 + \beta_3 V^3 + \beta_4 V^4 + \ldots\ldots \tag{24}$$

satisfying the initial conditions $U = 3$, $V = 0$. The coefficients $\beta_1, \beta_2, \beta_3$ and β_4 may be determined by straightforward means.

Thus, we have

$$\beta_1 = -\frac{3}{5}W, \quad \beta_2 = -\frac{1}{7}\beta_1(2\beta_1 + 1),$$

$$\beta_3 = \frac{1}{9}\beta_2(W - 5\beta_1 - 2),$$

$$\beta_4 = \frac{1}{11}\left[\beta_3(2W - 6\beta_1 - 3) - 3\beta_2^2\right].$$

Proceeding as in the foregoing section 3, we obtain the desired approximate analytical solution of (16) in the form of a rational function (Padé (2, 2) approximation) :

$$U(2,2) = 3\,\frac{(1 + A''V + B''V^2)}{(1 + C''V + D''V^2)}, \tag{25}$$

where $\quad A'' = C'' + \frac{1}{3}\beta_1, \qquad B'' = \frac{1}{3}(\beta_1 C'' + \beta_2) + D'',$

$$C'' = (\beta_1\beta_4 - \beta_2\beta_3)/\Delta, \qquad D'' = (\beta_3^2 - \beta_2\beta_4)/\Delta,$$

$$\Delta \equiv \beta_2^2 - \beta_1\beta_3$$

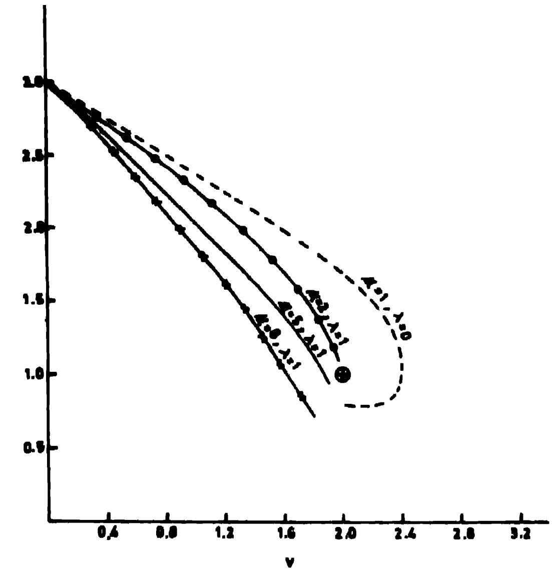

Figure 9. Characteristic features of non-rotating two-component isothermal spheres ($\lambda = 1, \mu = 2, W = 1.11$), ($\lambda = 1, \mu = 5, W = 1.44$) and ($\lambda = 1, \mu = 8, W = 1.60$). The dashed curve (- - - -) represent one-component case ($\lambda = 0, \mu = 1, W = 1$).

(U, V) curves (Figure 9) represent characteristic features of the three representative two-component ($\lambda = 1, \mu = 2, W = 1.11$), ($\lambda = 1, \mu = 5, W = 1.44$), ($\lambda = 1, \mu = 8, W = 1.60$) and one-component ($\lambda = 0, \mu = 1, W = 1$) [4] isothermal spheres. Figure 9 shows the fastest (monotonic) fall off in U (for assigned values of V) for the case of isothermal sphere defined by ($\lambda = 1, \mu = 8, W = 1.60$) than others.

(c) Distribution of mass, pressure and density inside the isothermal spheres :

Expression for the mass $M(r)$ interior to r (or ξ) can be written as

$$M(r) = \int_{o} 4\pi \rho r^2 dr = 4\pi\alpha^{-3} m_1 n_1(o)\, \xi^2 \frac{d\psi}{d\xi} \tag{26}$$

The pressure P interior to r (or ξ) is given by

$$P = K_B T\, [n_1(r) + n_2(r)], \tag{27}$$

which, by virtue of eqs. (4), (5) and (7) becomes

$$P = K_B T n_1(o)\, [e^{-\psi} + \lambda\mu e^{-\mu\psi}]. \tag{27'}$$

The density distribution ρ at a point r (or ξ) defined by

$$\rho = m_1 n_1(o) \exp \frac{m_1}{K_B T} [\varphi(r) - \varphi(o)] + m_2 n_2(o) \exp \frac{m_2}{K_B T} [\varphi(r) - \varphi(o)], \tag{28}$$

or with the help of eqs. (5) and (7), we have

$$\rho = m_1 n_1(\text{o})\, [e^{-\psi} + \lambda \mu e^{-\mu\psi}]. \tag{28'}$$

The mean density ($\bar{\rho}(r)$) is given by

$$\bar{\rho}(\xi) = M(r)/\frac{4}{3}\pi\xi^3\alpha^{-3} = m_1 n_1(\text{o})\, \frac{3}{\xi}\, \frac{d\psi}{d\xi}\,: \tag{29}$$

(d) The total energy :

Expression for the total number N_1 and N_2 of particles, in dimensionless forms are, respectively, given by

$$N_1 = 4\pi \int_0^\xi \rho_1\, \xi^2 \alpha^{-3}\, d\xi = 4\pi\alpha^{-3} n_1(\text{o}) \int_0^\xi \xi^2 e^{-\psi}\, d\xi, \tag{30}$$

and

$$N_2 = 4\pi \int_0^\xi \rho_2\, \xi^2 \alpha^{-3}\, d\xi = 4\pi\alpha^{-3} n_1(\text{o})\lambda \int_0^\xi \xi^2 e^{-\mu\psi}\, d\xi. \tag{31}$$

Addition of the expressions (16), (17) and the gravitational potential energy Ω yields the total energy E of the system

$$E = \frac{3}{2} K_B T\, (N_1 + N_2) + \Omega. \tag{32}$$

We note some more important physical implications of our present analytical technique as applied to the equilibrium eq. (6) : Any desired physical parameter of the stellar models, such as, $\rho_1(r)/\rho_1(\text{o})$ and $\rho_2(r)/\rho_1(\text{o})$, etc as shown in Figures 1-8, can be immediately obtained without computer programs. $\rho_1(r)/\rho_1(\text{o})$ and $\rho_2(r)/\rho_1(\text{o})$ measure the central concentrations of light and heavy particles, respectively, inside the configurations. For $\lambda' = 0$, the first term in eq. (6) dominates (light particles) leading to the well-known Chandrasekhar's assymptotic solution, at large ξ, for the one-component case. For $\lambda > 0$, the second term (eq. (6)) dominates (heavy particles). For small ξ, an expansion of the form (9) for the gravitational potential ψ holds for all λ. The heavy particles sink to the centre while the light particles are displaced outward. Figure 1 shows a comparative study of the mass concentrations of two-component and one-component configurations. In Figures 3, 4 and 5 are shown, respectively, pressure, density and mean density distributions inside the configurations. Characteristic features of the variations of $\xi^2\psi'$ (a quantity proportional to mass), ψ' (slope of the gravitational potential ψ), and ψ and are given in Figures 6, 7 and 8, respectively. A comparative study of the variations of U (product of $r/M(r)$ and rate of change of mass, $dM(r)/dr$) with V (product of $r/P(r)$ and rate of change of pressure $d\bar{p}(r)/dr$) between two component and one-component stellar models is shown in Figure 9.

Acknowledgments

The authors are grateful to the Council of Science and Technology, UP, Lucknow for financial support under the grants for the Research Project.

References

[1] R Emden 1907 *Gaskugeln* (Leipzig, B G Teubner)

[2] A S Eddington 1926 *The Internal Constitution of the Stars* (London : Cambridge University Press) p 79

[3] E A Milne 1930 *Handbook der Astrophys* **3** 25

[4] S Chandrasekhar 1939 *An Introduction to the Study of Stellar Structure* (Chicago : University of Chicago Press)

[5] L Spitzer 1942 *Astrophys J.* **95** 329

[6] P Leadoux 1951 *Ann D' Astrophys* **14** 438

[7] P Goldreich and D Lynden-Bell 1965 *M.N.R.A.S.* **130** 97

[8] J Ostriker 1964 *Astrophys J.* **140** 1056

[9] J Ostriker 1965 *Astrophys J. Suppl.* **11** 167

[10] E R Harrison and R G Lake 1972 *Astrophys J.* **171** 323

[11] J P Sharma 1981a *Indian J. Pure Appl. Maths.* **12** 119

[12] J P Sharma 1981b *DSc Thesis* (University of Gorakhpur, UP, India)

[13] L G Taff, H M Van Horn, C J Hansen and R R Ross 1975 *Astrophys J.* **197** 651

[14] A P Lightman 1977 *Astrophys J* **215** 914

[15] T W Edwards 1981a *Astrophys J.* **244** 619

[16] T W Edwards 1981b *Astrophys J. Suppl* **47** 291

[17] W Y Chau, K Lake and J Stone 1984 *Astrophys J.* **281** 560

[18] W Charles, Misner, Kips S Thame and J A Wheeler 1973 *Gravitation* (San Francisco : W H Freeman and Company)

[19] L Spitzer and M H Hart 1971 *Astrophys J.* **166** 483

[20] L Spitzer 1969 *Astrophys J. Lett.* **158** L139

[21] D Lynden-Bell and R Wood 1968 *M.N.R.A.S* **138** 495